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CREEP BUCKLING CONSIDERING MATERIAL DAMAGE

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Abstract—The influence of Kachanov–Rabotnov type damage on creep buckling of a compressed column is investigated. All deformation and damage of the column is assumed to take place at a "creep and damage hinge". Instantaneous buckling as well as creep buckling are shown to be represented by a certain instability surface in strain-damge-load space. The special case of purely brittle, i.e. non deforming, compressive instability is studied in some detail.

1. INTRODUCTION

Two phenomena characterize a material under creep conditions: the development of strain and of damage, both increasing with time under constant applied load. Relations between strain, stress and time are usually known as creep laws and relations between damage, stress and time as damage laws. Although recent microscopic analyses of strain and damage in engineering materials have given much insight into the physics of creep, phenomenological description is still preferred in analyzing macroscopic structural behaviour, see Odqvist and Hult[1], Hult[2], Rabotnov[3] and Penny and Marriott[4].

The bulk of structural creep analyses reported in literature concerns the idealized case without any damage creation. Expressions for lifetime, defined by criteria of maximum stress or maximum deformation, have been derived for various structural elements subject to various loading histories.

It is only to be expected that simultaneous presence of damage creation will lead to shorter lifetimes than those derived for non damaging behaviour. It is the purpose of this paper to examine this reduction in the case of creep buckling of columns.

The basic creep law and damage law to be employed here will first be stated and discussed in some detail. The main results of the theory of non damaging creep buckling will then be reviewed. A simplified structural model will be presented, which incorporates the main features of a column subject to both creep strain and creep damage. Finally the behaviour of this model will be analyzed and discussed.

2. CREEP LAWS AND DAMAGE LAWS

Stationary creep analyses are usually based on a law of stationary creep having the form

$$\frac{d\epsilon}{dt} = F(\sigma). \tag{1}$$

Here ϵ denotes strain, σ stress, t time and F a usually nonlinear function, often taken as a simple power function (Norton's law). This law is established from creep strain measurements at early stages of constant load creep tests, where the stress is approximately constant in time. An extension of (1) to cases of time variable stress is often stated in the form

$$\frac{d\epsilon}{dt} = G'(\sigma)\frac{d\sigma}{dt} + F(\sigma). \tag{2}$$

The added term corresponds to an instantaneous stress–strain relation of the form

$$\epsilon = G(\sigma). \tag{3}$$

The creep law (2) is a generalization of a form first suggested by Odqvist [5], who took $G(\sigma)$ as a simple power function.

A damage law closely corresponding to the stationary creep law (1) was proposed by Kachanov [6] in the form

$$\frac{d\omega}{dt} = f(s). \tag{4}$$

Here ω denotes damage, and s is a net stress defined by

$$s = \frac{\sigma}{1-\omega}. \tag{5}$$

The function f was taken by Kachanov as a simple power function.

If damage is created also at sudden increases in the net stress, the damage law (4) may be extended in the same way as was done with the creep law (1), see Hult and Broberg [7], to yield

$$\frac{d\omega}{dt} = g'(s)\frac{ds}{dt} + f(s). \tag{6}$$

The added term corresponds to an instantaneous stress-damage relation of the form .

$$\omega = g(s). \tag{7}$$

The presence of damage will necessarily also affect the rate of creep strain, and the following generalization of (2) has been suggested, see Broberg [8]

$$\frac{d\epsilon}{dt} = G'(s)\frac{ds}{dt} + F(s). \tag{8}$$

If no damage is created, equation (8) reduces to equation (2).

If the term corresponding to instantaneous strain is deleted from equation (8), a creep law earlier suggested by Rabotnov [3] is obtained.

The behaviour of a tensile bar subject to an instantaneously applied constant load and obeying the damage and creep laws (6) and (8) has been examined by Broberg [8]. The main results may be summarized as follows.

(a) Rupture may occur already during load application, before the intended constant load level is reached. Under certain conditions $d\epsilon/dP \to \infty$ and $d\omega/P \to \infty$ at some load $P = P_R$. The corresponding strain $\epsilon = \epsilon_R$ and damage $\omega = \omega_R$ are finite quantities ($\epsilon_R < \infty$, $\omega_R < 1$), i.e. both elongation and damage are of limited magnitude in the ruptured bar. For non damaging, i.e. completely ductile materials such instantaneous tensile instability was analysed already by Considère[9].

(b) If instantaneous rupture does not occur, i.e. if the applied load is smaller than the rupture load P_R, rupture will occur after a certain time under load. It is found that $d\epsilon/dt \to \infty$ and $d\omega/dt \to \infty$ at a finite time $t = t_*$. The corresponding strain $\epsilon = \epsilon_*$ and damage $\omega = \omega_*$ are still finite quantities ($\epsilon_* < \infty$, $\omega_* < 1$). For non damaging materials such delayed tensile instability, usually denoted ductile creep rupture, was first analysed by Hoff[10]. For non deforming, i.e. completely brittle, materials obeying the damage law (4), a creep rupture lifetime was derived by Kachanov[6]. He also considered a simplified case of mixed ductile–brittle creep rupture.

(c) The two kinds of tensile rupture, viz. instantaneous and delayed, may be given a common interpretation.

The state of the bar is defined by the three variables P, ϵ and ω, and hence may be represented by a point in a P, ϵ, ω-space. The unloaded, unstrained, undamaged state corresponds to the origin. Upon loading the state point will move out in the first octant, away from the origin. Instability, i.e. tensile rupture, will occur when the point reaches a certain surface, convex towards the origin and fixed in the P, ϵ, ω-space (Fig. 1a). Instantaneous load application

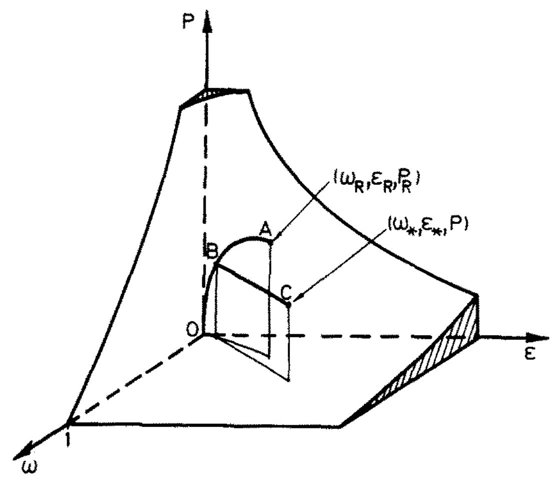

Fig. 1a. Instability surface for tensile case.

corresponds to the path OBA, where A is the instability point, located on the instability surface, corresponding to instantaneous rupture. If the load increase is terminated at B, and the load P is kept constant, the path BC will be followed. Here C is located on the instability surface and corresponds to creep rupture. For the two extreme cases of purely ductile ($\omega \equiv 0$) and purely brittle ($\epsilon \equiv 0$) behaviour the corresponding paths are shown in Fig. 1b, marked with ′ and ″ respectively. The instability surface then corresponds to instability curves, marked as shaded curved lines. The existence of such a common instability curve for the purely ductile case was earlier noted by Carlson[11].

In the sequel the behaviour of a compressed, slightly curved, column will be examined under similar conditions. Strong similarities with the tensile instability phenomena will be brought out.

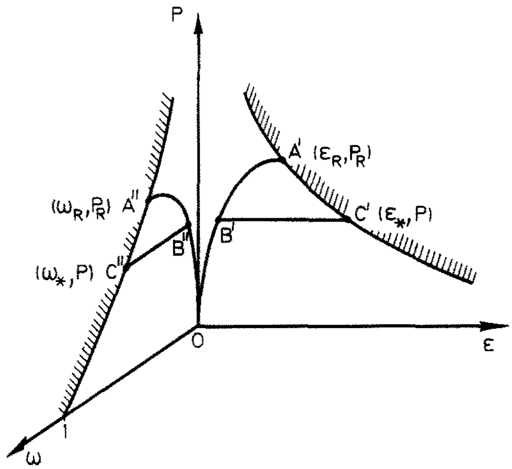

Fig. 1b. Instability curves for purely ductile (′) and purely brittle (″) tensile cases.

3. CHARACTERISTICS OF NON DAMAGING, DUCTILE CREEP BUCKLING

Two rather different approaches to the problem of creep buckling have been proposed:

(a) Rabotnov and Shesterikov[12] considered the stability of an axially loaded straight column subject to ductile creep deformation. Including inertial mass in the analysis they formulated a criterion for the column to be stable after a certain time under load, considering the response to an infinitesimal disturbance of the straight equilibrium form. Linearization allowed a critical time to be determined after which the column ceased to be stable.

(b) Hoff[13, 14] analysed the deformation of an initially slightly curved column subject to ductile creep deformation under constant axial load. Inertial effects were disregarded. A critical time was determined after which the rate of deflection became infinite.

These two approaches to the ductile creep buckling problem have been compared mainly with respect to applicability in design work (see Hoff[15, 16]). Although the Rabotnov–Shesterikov analysis is of strong conceptual interest, the Hoff type analysis has become predominant in application oriented studies. One main reason for this is that initial imperfections, which form a basis of the Hoff approach and which have a decisive influence on the critical time, are always present in real columns.

For the same reason the present study of creep buckling with damage will deal with an initially curved column, and Hoff's main results for the non damaging case will therefore first be briefly reviewed.

The creep law was taken in the form

$$\frac{d\epsilon}{dt} = B_0\sigma^{n_0}\frac{d\sigma}{dt} + B\sigma^n; \qquad 1 < n_0 < n$$

see (2) above.

The first analysis dealt with statically determinate H type cross sections and the results were later extended to other cross sectional forms. The column was assumed to have a slight sinusoidal initial curvature, and to retain the sinusoidal form throughout the buckling process. The midpoint deflection δ was determined as a function of axial load P and time t. For step type loading the

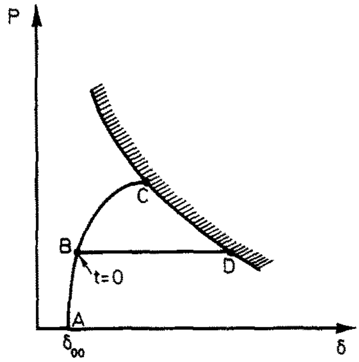

Fig. 2. Buckling of column. Shaded curve indicates instability. *ABC* shows instantaneous buckling and *ABD* creep buckling.

resulting deflection versus load is as shown in Fig. 2, where δ_{00} is the initial midspan deflection. On account of the nonlinear relation between stress and strain or strain rate the compressed column will not retain its initial form (see Fig. 3). Later studies applying Fourier series have therefore been made to find the true deflected form. Furthermore, the influence of deloading in part of the cross section has been examined, etc.

In the creep law used by Hoff elastic deformation was disregarded. Including a term for linear elastic deformation Hult[17] calculated a corresponding shortening of the time to buckling.

4. SIMPLIFIED COLUMN MODEL

For highly nonlinear dependence of strain and/or strain rate on stress the column in Fig. 3 will take a strongly pointed shape. Therefore, to simplify the analysis all deformation and damage will be assumed here to take place in one cross section only. For materials obeying the creep law $\dot{\epsilon} \sim \sigma^n$ this corresponds to the limiting case $n = \infty$. The concept of a creep hinge has previously been used, e.g. in analyses of creep buckling in frameworks[18]. Here it will be generalized to include also damage creation.

Consider a column consisting of two rigid parts joined by a "creep and damage hinge" according to Fig. 4. No bending moment can be transmitted through the ends of the column. The

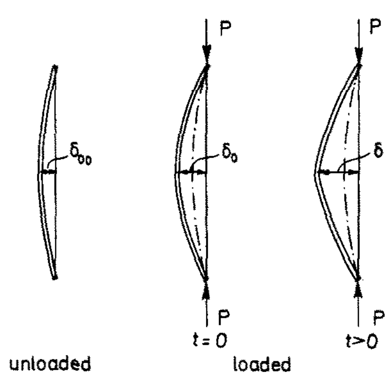

Fig. 3. Creep buckling of slightly curved column.

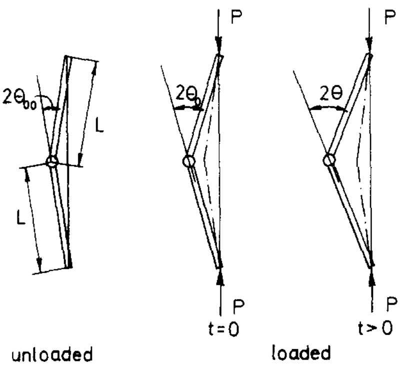

Fig. 4. Simplified column model.

angle Θ is assumed to be small. Then the bending moment at the hinge is

$$M = PL\Theta$$

where P is the compressive force, L is half the column length and Θ is the end point slope of the column. In analogy to equation (5) we define an effective hinge moment as

$$S = \frac{M}{1-\omega}$$

where ω is the damage. Hence

$$S = \frac{PL\Theta}{1-\omega}. \tag{9}$$

The rotation of the hinge, away from the initial state is described by the one variable

$$2\theta = 2\Theta - 2\Theta_{00}. \tag{10}$$

In analogy to equations (8) and (6) the following deformation and damage laws for the hinge will be assumed:

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = G'(S)\frac{\mathrm{d}S}{\mathrm{d}t} + F(S)$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = g'(S)\frac{\mathrm{d}S}{\mathrm{d}t} + f(S).$$

This hinge model does not take into account the differences in tensile and compressive behaviour. From earlier studies of ductile creep buckling it is concluded that such a simplification does not appreciably affect the behaviour of the column.

Considering equation (10) these will be written as

$$\mathrm{d}\Theta = G'(S)\,\mathrm{d}S + F(S)\,\mathrm{d}t \tag{11}$$

$$\mathrm{d}\omega = g'(S)\,\mathrm{d}S + f(S)\,\mathrm{d}t. \tag{12}$$

Relations (9), (11) and (12) constitute the governing equations for the column model. They will be analysed for the case of step loading

$$P = P_0 H(t)$$

where $H(t)$ is the Heaviside unit step function.

5. CREEP BUCKLING ANALYSIS

The initial conditions for $\Theta(t)$ and $\omega(t)$ are

$$\Theta(0^-) = \Theta_\infty$$
$$\omega(0^-) = 0.$$

The development of $\Theta(t)$ will be studied first during load application ($0^- < t < 0^+$) and then for the creep process ($t > 0^+$). Conditions for instability will be established for both these phases.

Eliminating formally dS and $d\omega$ from equations (9), (11) and (12) we obtain

$$d\Theta\{PL\Theta - S^2 g'(S) - SPLG'(S)\} = SL\Theta G'(S)\, dP + \{PL\Theta F(S) - S^2[F(S)g'(S) - f(S)G'(S)]\}\, dt. \tag{13}$$

If on the other hand dS and $d\Theta$ are eliminated we have

$$d\omega\{PL\Theta - S^2 g'(S) - SPLG'(S)\} = SL\Theta g'(S)\, dP + \{PL\Theta f(S) + PLS[F(S)g'(S) - f(S)G'(S)]\}\, dt. \tag{14}$$

Time interval $0^- < t < 0^+$

Equations (13) and (14) show that $(d\Theta/dP) \to \infty$ and $(d\omega/dP) \to \infty$ when $PL\Theta - S^2 g'(S) - SPLG'(S) \to 0$. We define this to be a situation of instantaneous buckling (mixed ductile and brittle).

Time interval $t > 0^+$

From equations (13) and (14) we have $(d\Theta/dt) \to \infty$ and $(d\omega/dt) \to \infty$ when $PL\Theta - S^2 g'(S) - SPLG'(S) \to 0$. This will be referred to as a situation of creep buckling (mixed ductile and brittle).

Hence a common condition for both kinds of buckling exists, viz.

$$P_* L\Theta_* - S_*^2 g'(S_*) - S_* P_* LG'(S_*) = 0 \tag{15}$$

or considering equation (9)

$$\frac{P_* L\Theta_*}{1-\omega_*}\left\{\frac{1}{1-\omega_*} g'\left(\frac{P_* L\Theta_*}{1-\omega_*}\right) + \frac{1}{\Theta_*} G'\left(\frac{P_* L\Theta_*}{1-\omega_*}\right)\right\} = 1 \tag{16}$$

where the subscript * indicates instability. It is necessary that $\omega_* < 1$ and that Θ_* is finite except for some trivial cases. The functions f, F, G', g' must be finite in the corresponding S-interval. Note that P not needs to be constant. If P is a continuous function of t the statement about $d\Theta/dt$ and $d\omega/dt$ still holds. When P is discontinuous, the instantaneous buckling concept applies.

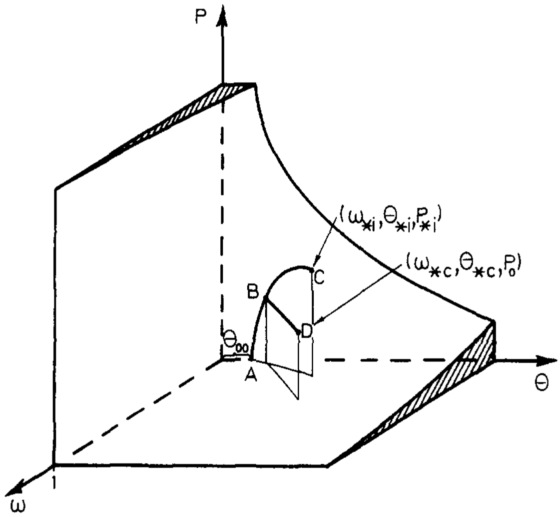

Fig. 5. Instability surface for compressive case.

Equation (16) defines an instability surface in the P, Θ, ω-space, see Fig. 5. In Fig. 5 the path ABC denotes the development of Θ and ω for instantaneously increasing P. Buckling will occur at C when $P = P_{*i}$, $\Theta = \Theta_{*i}$ and $\omega = \omega_{*i}$. If $P_0 < P_{*i}$ the path ABD will be followed. Creep buckling will occur at D when $\Theta = \Theta_{*c}$ and $\omega = \omega_{*c}$.

The time to reach creep buckling may be determined from equations (13) and (14) which, for $P = P_0 = \text{const}$, take the forms

$$\frac{\mathrm{d}\Theta}{\mathrm{d}t} = K(P_0, \Theta, \omega) \tag{17}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = k(P_0, \Theta, \omega). \tag{18}$$

Step by step integration maps out the path BD in the P_0, Θ, ω-plane. The creep buckling time is obtained from the simultaneous fulfilment of the conditions $\mathrm{d}\Theta/\mathrm{d}t \to \infty$ and $\mathrm{d}\omega/\mathrm{d}t \to \infty$. The

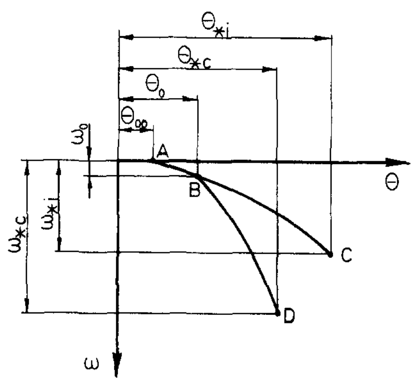

Fig. 6. Development of Θ and ω during instantaneous and creep loading.

starting values $\Theta = \Theta_0$ and $\omega = \omega_0$ (see Fig. 6) for the integration are taken from the analyis of the instantaneous loading.

Numerical calculations will not be undertaken here, since experimental results are yet very meager in this area.

6. DISCUSSION

The two extreme cases of purely ductile instability ($\omega \equiv 0$) and purely brittle instability ($\theta \equiv 0$) are of interest and will be studied in some detail.

Ductile instability ($\omega \equiv 0$)

If the functions F and G' are taken as power functions equations (9) and (11) yield the same result as found previously by Hoff.

This case corresponds fully to the one of ductile tensile instability[11].

Brittle instability ($\theta \equiv 0$)

The functions f and g are chosen as simple power functions, i.e.

$$g(S) = C_0 S^{\nu_0}$$

$$f(S) = CS^{\nu}.$$

Then equations (9) and (12) take the forms

$$S = \frac{PL\Theta_{00}}{1-\omega} \tag{19}$$

$$d\omega = \nu_0 C_0 S^{\nu_0 - 1}\, dS + CS^{\nu}\, dt. \tag{20}$$

Load application. By integration of equation (20) we have for the time interval $0^- < t < 0^+$

$$\omega = C_0 S^{\nu_0}$$

or considering (19)

$$\omega(1-\omega)^{\nu_0} = C_0(PL\Theta_{00})^{\nu_0}. \tag{21}$$

The instability condition $d\omega/dP = \infty$ at $\omega = \omega_{*ib}$ and $P = P_{*ib}$ yields with equation (21)

$$\omega_{*ib} = \frac{1}{1+\nu_0} \tag{22}$$

$$P_{*ib} = \frac{\nu_0}{(1+\nu_0)^{(1+\nu_0)/\nu_0}} \cdot \frac{1}{(C_0^{(1/\nu_0)} L\Theta_{00})}. \tag{23}$$

Constant load. With $P = P_0 < P_{*ib}$ follows from equations (19) and (20)

$$Cm^{\nu}t = \frac{\nu_0 C_0 m^{\nu_0}}{\nu - \nu_0}(1-\omega)^{\nu-\nu_0} - \frac{(1-\omega)^{\nu+1}}{\nu+1} - \frac{\nu_0 C_0 m^{\nu_0}}{\nu-\nu_0}(1-\omega_0)^{\nu-\nu_0} + \frac{(1-\omega_0)^{\nu+1}}{\nu+1} \tag{24}$$

where $m = P_0 L \Theta_{00}$. This gives $d\omega/dt \to \infty$ at $\omega = \omega_{*b}$, where

$$(1-\omega_{*b})^{\nu} - \nu_0 C_0 m^{\nu_0}(1-\omega_{*b})^{\nu-\nu_0-1} = 0$$

from which follows

$$(1-\omega_{*b})^{\nu_0+1} = \nu_0 C_0 m^{\nu_0}. \tag{25}$$

From (24) is then found the corresponding instability time t_{*b}, given by

$$Cm^{\nu}t_{*b} = \frac{1+\nu_0}{(1+\nu)(\nu-\nu_0)}(\nu_0 C_0 m^{\nu_0})^{(1+\nu)/(1+\nu_0)} + \frac{1}{1+\nu}(1-\omega_0)^{1+\nu} - \frac{\nu_0 C_0 m^{\nu_0}}{\nu-\nu_0}(1-\omega_0)^{\nu-\nu_0}. \tag{26}$$

For any given load $P_0 < P_{*ib}$ and initial deflection Θ_{00} the corresponding instantaneous damage ω_0, resulting from load application, may be found from equation (21). The instability time t_{*b} is then obtained from equation (26).

These results may be summarized in a diagram as shown by Fig. 7. The close similarity with Fig. 1b, referring to purely brittle tensile rupture, is evident.

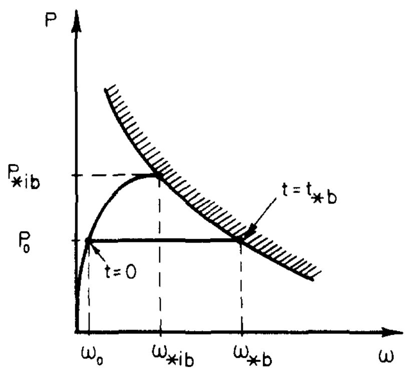

Fig. 7. Instability curve for purely brittle case.

For materials without instantaneous damage creation $C_0 \equiv 0$, and hence (26) degenerates to

$$Cm^{\nu}t_{*b} = \frac{1}{1+\nu}(1-\omega_0)^{1+\nu}.$$

For an initially completely undamaged material $\omega_0 = 0$, and hence

$$Cm^{\nu}t_{*b} = \frac{1}{1+\nu}.$$

This expression is fully equivalent to the expression for tensile, brittle rupture lifetime t_* derived by Kachanov [6]

$$C\sigma^{\nu}t_* = \frac{1}{1+\nu}.$$

In a similar way a close correspondence exists between tensile and compressive instability also in the more general situation, when both ductile deformation and brittle damage occur simultaneously. This is clearly shown by the strong similarity between Figs. 1a and 5.

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